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Stress tensor of a strained material with a linear row of stress concentrators

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Abstract

An approach for the evaluation of stress tensors in a strained material with a linear row of stress concentrators (i.e. voids, gas bubbles and secondary phase precipitates in material science) is discussed. The technique can be applied for materials with concentrators whose form can be conformally mapped onto the unit circle by a rational function and ranges over all application-important types of applied loadings within the approximation of plane strain. The approach is applied to the computation of stress tensors in material subjected to uniaxial loading, uniform stress and simple shear. The effect that a row of voids has on local stress field redistribution in a loaded material is investigated.

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1. Introduction

Voids, gas bubbles, secondary phase precipitates etc. are common feature of the microstructure of modern structural reactor materials. These inhomogenities are regarded as stress concentrators in fracture theory.

Stress concentrators can noticeably modify physical–mechanical properties of structural materials. It is known that these changes can be both positive and negative. Moreover, the influence can be varied from a positive to a negative one with an external stress field alteration. An example of a positive effect of stress concentrators is strengthening of structural material subjected to thermal treatment (heat ageing) in order to form fine-dispersed secondary phase precipitates. A negative example is helium bubbles and void formation in materials under irradiation, namely high temperature embrittlement of structural reactor materials.

A number of relevant analytical approaches deal with an analysis of stress field around stress concentrators [1–6] and their arrays [7–11]. Investigations [1–6] were carried out under assumption of negligible influence on the part of neighbor concentrators. The assumption is fulfilled provided the distance between stress concentrators is at least an order of magnitude larger than the size of the stress concentrator. Otherwise the effect of the neighborhood of a stress concentrator should be taken into account. Quantitative considerations have been carried out; however, they were either too complicated for practical application in materials science [7–10] or submitted several particular cases only [7,11].

In this paper the calculation of stress tensor $\sigma_{ij} = \sigma_{ij}(r, \theta)$ in a strained material with a linear array of stress concentrators is carried out through the technique proposed in [11] with some modifications.

2. Problem formulation

In order to calculate the stress tensor, let us consider an infinite plane with a linear row of uniform stress concentrators (see Fig. 1). Centers of the stress concentrators lie on the Ox axis of the Cartesian coordinate system. The plane is loaded with an external stress tensor σ. The distance between centers of neighbor

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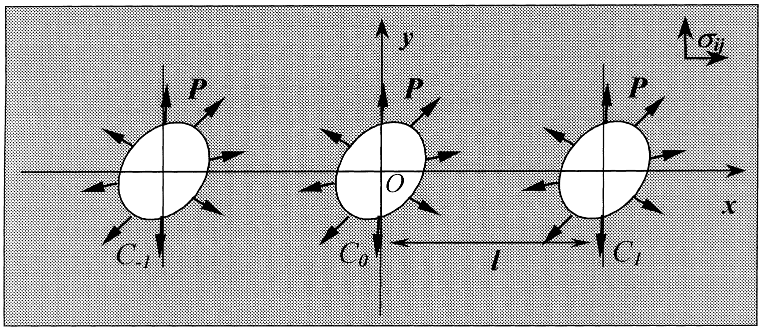

Fig. 1. Geometry of the problem.

concentrators in the row is l. Internal surfaces of the concentrators are subjected to normal traction P. Possible plasticity effects in the vicinity of the stress concentrators are neglected in the framework of current consideration. The center, O, of the coordinate system is chosen in the center of a concentrator in the row. Its shape is denoted as C_0. The shapes of other concentrators are marked as C_k, $k = \pm 1, \pm 2, \ldots, \pm\infty$ (see Fig. 1). We consider the case of plane strain.

3. Governing equations

The problem is treated in terms of the complex potential approach [9], where the plane (x, y) is described through the complex quantity $z = x + \mathrm{i}y$.

According to [9–11] the complex potentials φ and ψ for the elastic problem under consideration can be written as

$$\varphi(z) = A(z) + \varphi_0(z), \tag{1}$$

$$\psi(z) = B(z) - z\frac{\mathrm{d}A(z)}{\mathrm{d}z} + \psi_0(z), \tag{2}$$

where $A(z)$ and $B(z)$ are the functions holomorphic outside the region bounded by C_k, $k = 0 \pm 1, \pm 2, \ldots, \infty$, $\varphi_0(z)$ and $\psi_0(z)$ are the complex potentials describing the strain of the plane without stress concentrators. Due to the geometry of the problem both $A(z)$ and $B(z)$ are functions periodic on x with a period l. According to the general approaches of complex analysis [11] and the mathematical theory of elasticity [9,12], in the region $|\xi - z| < l$ $A(z)$ and $B(z)$ can be obtained in the form

$$A(z) = K(z) - \frac{1}{\pi\mathrm{i}}\sum_{k=0}^{\infty}\frac{\alpha_{2k+2}}{l^{2k+2}}\int_{C_0} K(\xi)(\xi - z)^{2k+1}\,\mathrm{d}\xi, \tag{3}$$

$$B(z) = L(z) - \frac{1}{\pi\mathrm{i}}\sum_{k=0}^{\infty}\frac{\alpha_{2k+2}}{l^{2k+2}}\int_{C_0} L(\xi)(\xi - z)^{2k+1}\,\mathrm{d}\xi, \tag{4}$$

where $K(z)$ and $L(z)$ are the functions holomorphic everywhere outside C_0, and

$$\alpha_k = \sum_{n=1}^{\infty}\frac{1}{n^k}. \tag{5}$$

Applying the complex potentials φ and ψ (Eqs. (1) and (2)), taking account of Eqs. (3) and (4) to the boundary conditions of the first problem of the theory of elasticity [9] yields

$$\begin{aligned} f_1(t) + \mathrm{i}f_2(t) &= K(t) + \overline{\frac{\mathrm{d}K(t)}{\mathrm{d}t}}(t - \bar{t}) + \overline{L(t)} \\ &\quad - \frac{1}{\pi\mathrm{i}}\sum_{k=0}^{\infty}\frac{\alpha_{2k+2}}{l^{2k+2}}\int_{C_0} K(\xi)(\xi - t)^{2k+1}\,\mathrm{d}\xi \\ &\quad + \frac{1}{\pi\mathrm{i}}\sum_{k=0}^{\infty}\frac{\alpha_{2k+2}}{l^{2k+2}}\int_{C_0}\overline{L(\xi)(\xi - t)^{2k+1}\,\mathrm{d}\xi} \\ &\quad - \frac{(t - \bar{t})}{\pi\mathrm{i}}\sum_{k=0}^{\infty}\frac{\alpha_{2k+2}}{l^{2k+2}}(2k+1)\int_{C_0}\overline{K(\xi)(\xi - t)^{2k}\,\mathrm{d}\xi} + \varphi_0(t) \\ &\quad + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)}. \end{aligned} \tag{6}$$

Here $\overline{K(t)}$, $\overline{L(t)}$, $\bar{t}$, $\overline{\varphi_0'(t)}$ and $\overline{\psi_0(t)}$ denote the complex conjugate values of $K(t)$, $L(t)$, t, $\varphi_0'(t)$ and $\psi_0(z)$, respectively; t is the complex value on the contour C_0 and the sum $f_1(t) + \mathrm{i}f_2(t)$ is calculated from the following relation:

$$f_1(t) + \mathrm{i}f_2(t) = \mathrm{i}\int_{t_c}^{t}(p_x + \mathrm{i}p_y)\,\mathrm{d}s, \text{ on the contour } C_0, \tag{7}$$

where $p_k = \sigma_{kj}|C_0 \cdot \cos(\vec{n}, \ \vec{e_j}), (j, k = x, y)$, is the resultant force acting at the contour $C_0, \vec{n}$ is a unit vector of the outward normal to the contour C_0, $\vec{e_j}$ $(j = x, y)$ are the unit basis vectors, t_c is the arbitrarily fixed point on the contour C_0. The path-tracing is designated positive provided the plane leaves in the left.

Now let us expand functions $K(z)$ and $L(z)$ into a series over $1/l^2$

$$K(z) = \sum_{m=0}^{\infty} \frac{1}{l^{2m}} K_{2m}(z), \tag{8}$$

$$L(z) = \sum_{m=0}^{\infty} \frac{1}{l^{2m}} L_{2m}(z). \tag{9}$$

Substitution of expansions (8) and (9) into the boundary condition (6) and separation of terms with equal orders of $1/l^2$ results in the following equation set:

$$\begin{aligned} K_0(t) &= \overline{\frac{\mathrm{d}K_0(t)}{\mathrm{d}t}}(t-\bar{t}) + \overline{L_0(t)} - \varphi_0(t) - \overline{t\varphi_0'(t)} - \overline{\psi_0(t)} \\ &\quad + f_1 + \mathrm{i} f_2, \\ K_2(t) &= \overline{\frac{\mathrm{d}K_2(t)}{\mathrm{d}t}}(t-\bar{t}) + \overline{L_2(t)} + \frac{\alpha_2}{\pi \mathrm{i}} \int_{C_0} K_0(\xi)(\xi - t)\,\mathrm{d}\xi \\ &\quad - \frac{\alpha_2}{\pi \mathrm{i}} \int_{C_0} \overline{L_0(\xi)(\xi - t)\,\mathrm{d}\xi} + \frac{\alpha_2 (t-\bar{t})}{\pi \mathrm{i}} \int_{C_0} \overline{K_0(\xi)\,\mathrm{d}\xi} \\ &\vdots \\ K_{2m}(t) &= \overline{\frac{\mathrm{d}K_{2m}(t)}{\mathrm{d}t}}(t-\bar{t}) + \overline{L_{2m}(t)} \\ &\quad - \frac{1}{\pi \mathrm{i}} \sum_{k=0}^{m=1} \alpha_{2k+2} \int_{C_0} K_{2m-2k-2}(\xi)(\xi - t)^{2k+1}\,\mathrm{d}\xi \\ &\quad - \frac{1}{\pi \mathrm{i}} \sum_{k=0}^{m=1} \alpha_{2k+2} \int_{C_0} \overline{L_{2m-2k-2}(\xi)(\xi - t)^{2k+1}\,\mathrm{d}\xi} \\ &\quad + \frac{(t-\bar{t})}{\pi \mathrm{i}} \sum_{k=0}^{\infty} \alpha_{2k+2}(2k+1) \int_{C_0} \overline{K_{2m-2k+2}(\xi)(\xi - t)^{2k}\,\mathrm{d}\xi}. \end{aligned} \tag{10}$$

So, we obtain the equation set where the successive equation (i.e., the successive term in expansions (8) and (9)) is defined in terms of the previous ones.

Because the discussed approach is aimed at obtaining particular expressions for the evaluation of stress tensor components, we do not discuss the problem of convergence of (8) and (9). But we can state that the coefficients (10) of the expansions (8) and (9) can be obtained and these expansions converge as well for the concentrators whose shape conformally mapped onto the unit circle by a rational function. Detailed consideration can be found in [9].

The solution of the first boundary problem of the elasticity theory for a material with a linear row of stress concentrators is given by

$$\begin{aligned} \varphi(z) &= K_0(z) + \sum_{m=1}^{\infty} \frac{1}{l^{2m}} K_{2m}(z) - \frac{1}{\pi \mathrm{i}} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha_{2k+2}}{l^{2k+2m+2}} \\ &\quad \times \int_{C_0} K_{2m}(z)(\xi - z)^{2k+1}\,\mathrm{d}\xi + \varphi_0(z), \end{aligned} \tag{11}$$

$$\begin{aligned} \psi(z) &= L_0(z) + \psi_0(z) + \sum_{m=1}^{\infty} \frac{1}{l^{2m}} L_{2m}(z) \\ &\quad - \frac{1}{\pi \mathrm{i}} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha_{2k+2}}{l^{2k+2m+2}} \int_{C_0} L_{2m}(z)(\xi - z)^{2k+1}\,\mathrm{d}\xi \\ &\quad - z\frac{\mathrm{d}K_0(z)}{\mathrm{d}z} - z\frac{z}{\mathrm{d}z}\left(\sum_{m=1}^{\infty} \frac{1}{l^{2m}} K_{2m}(z) \right) \\ &\quad - z\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{\pi \mathrm{i}} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha_{2k+2}}{l^{2k+2m+2}} \int_{C_0} K_{2m}(z)(\xi - z)^{2k+1} \right) \mathrm{d}\xi, \end{aligned} \tag{12}$$

provided the equation set (10) can be solved.

Let us consider the first equation in set (10). It can be solved in the conventional way (see e.g., [9,11,13]). However, very often a simpler approach can be applied. Let us evaluate $\lim_{l\to\infty} \varphi(z)$ and $\lim_{l\to\infty} \psi(z)$. According to Eqs. (11) and (12)

$$\lim_{l\to\infty} \varphi(z) \equiv \varphi_\infty(z) = K_0(z) + \varphi_0(z), \tag{13}$$

and

$$\lim_{l\to\infty} \psi(z) \equiv \psi_\infty(z) = L_0(z) - z\frac{\mathrm{d}K_0(z)}{\mathrm{d}z} + \psi_0(z). \tag{14}$$

The condition $l \to \infty$ reduces the problem to the limiting case of the isolated stress concentrator in the strained plane. A lot of these problems either have been solved (see e.g. [1,2,6,9]) or the solution can be easily found through well developed techniques [9,13]. So, the problem under consideration can be solved based on the following algorithm:

1. Evaluation of the complex potentials $\varphi_\infty(z)$ and $\psi_\infty(z)$ for the isolated stress concentrator with the shape equivalent to that of the concentrators in the row in the material loaded with the same stress tensor, and determination of complex potentials $\varphi_0(z)$ and $\psi_0(z)$ describing the strain of the plane without stress concentrators.
2. Calculation of the zeroth order terms $K_0(z)$ and $L_0(z)$ of expansions (8) and (9) according to Eqs. (13) and (14).
3. Computation of the first and the following order terms $K_{2m}(z)$ and $L_{2m}(z)$ of expansions (8) and (9) by equation set (10) to an order sufficient to achieve the desired accuracy.
4. Determination of approximate expressions of the complex potentials $\varphi(z)$ and $\psi(z)$ via Eqs. (11) and (12).
5. Evaluation of the stress tensor through the Kolosov–Muskhelishvili equations [9,13]

$$\sigma_{rr} + \sigma_{\theta\theta} = 2\left(\varphi'(z) + \overline{\varphi'(z)}\right), \tag{15}$$

$$\sigma_{\theta\theta} - \sigma_{rr} + 2\mathrm{i}\sigma_{r\theta} = 2\left(\bar{z}\varphi''(z) + \psi'(z)\right). \tag{16}$$

4. Stress tensor

Let us carry out evaluation of the stress tensor in a strained plane with a linear row of circular stress concentrators. The centers of the concentrators with radius R are arranged on the Ox axis of the Cartesian coordinate system. The distance between the centers of neighbor concentrators is $l, 2R < l < \infty$.

In order to build stress tensors for a number of applications, relevant to cases of external and internal loadings, the problem is divided into two parts: (i) evaluation of the stress tensor of the plane without external loading with a linear row of stress concentrators with edges subjected to normal traction P (a row of dilatation centers) and (ii) calculation of stress tensor of a strained plain with a linear row of circular holes [7]. The plane is loaded uniaxially with a constant stress p which makes an angle ω with the Ox axis.

4.1. Stress tensor of a plane without external loading with a linear row of dilatation centers

An infinite plane contains an infinite row of circular stress concentrators with radius R. The distance between the neighbor concentrators is l. The internal surface of the stress concentrators is loaded with normal traction P.

In order to calculate the components of the stress tensor of a plane with the row of stress concentrators let us carry out calculations according to the algorithm proposed in Section 3.

1. The complex potentials $\varphi_\infty(z)$ and $\psi_\infty(z)$ for the isolated stress concentrator are given by [9]

$$\varphi_\infty(z) = 0, \quad \psi_\infty(z) = -\frac{PR^2}{z}. \tag{17}$$

   It is clear that both the complex potentials $\varphi_0(z)$ and $\psi_0(z)$ describing the strain of the externally unloaded plane without stress concentrators are equal to zero.
2. The zeroth order terms $K_0(z)$ and $L_0(z)$ of expansions (8) and (9) are

$$K_0(z) = 0, \quad L_0(z) = -\frac{PR^2}{z}. \tag{18}$$

3. The first order terms of expansions (8) and (9) can be written as

$$\begin{aligned} K_2(z) &= -\frac{2PR^4\alpha_2}{z}, \\ L_2(z) &= \frac{2PR^4\alpha_2}{z}\left(1 - \frac{R^2}{z^3}\right). \end{aligned} \tag{19}$$

4. Approximate expressions for the complex potentials $\varphi(z)$ and $\psi(z)$ are as follows:

$$\begin{aligned} \varphi(z) &\approx \varphi^{(0)}(z) + \varphi^{(1)}(z)(R/l)^2, \\ \psi(z) &\approx \psi^{(0)}(z) + \psi^{(1)}(z)(R/l)^2, \end{aligned} \tag{20}$$

   where

$$\begin{aligned} \varphi^{(0)}(z) &= 0, \\ \psi^{(0)}(z) &= -\frac{PR^2}{z}, \\ \varphi^{(1)}(z) &= -\frac{2\alpha_2 PR^2}{z}, \\ \psi^{(1)}(z) &= 2\alpha_2 Pz\left(1 - \frac{R^4}{z^4}\right). \end{aligned} \tag{21}$$

5. The approximate expression for the stress tensor $\sigma_{ij}(r,\theta), (i,j = r,\theta)$ is given by

$$\sigma_{ij}(r,\theta) \approx \sigma_{ij}^{(0)}(r,\theta) + \sigma_{ij}^{(1)}(r,\theta)\left(\frac{R}{l}\right)^2, \tag{22}$$

   where the zeroth order approximation

$$\begin{aligned} \sigma_{rr}^{(0)}(r,\theta) &= -\frac{PR^2}{r^2}, \\ \sigma_{\theta\theta}^{(0)}(r,\theta) &= \frac{PR^2}{r^2}, \\ \sigma_{r\theta}^{(0)}(r,\theta) &= 0, \end{aligned} \tag{23}$$

   corresponds to a dilatation center [9,13]. The first order term in expansion (22) is given by

$$\begin{aligned} \sigma_{rr}^{(1)}(r,\theta) &= -2\alpha_2 P\left(1 - 4\frac{R^2}{r^2} + 3\frac{R^4}{r^4}\right)\cos 2\theta, \\ \sigma_{\theta\theta}^{(1)}(r,\theta) &= 2\alpha_2 P\left(1 + \frac{3R^4}{r^4}\right)\cos 2\theta, \\ \sigma_{r\theta}^{(1)}(r,\theta) &= 2\alpha_2 P\left(1 + \frac{2R^2}{r^2} - \frac{3R^4}{r^4}\right)\sin 2\theta, \end{aligned} \tag{24}$$

   where $\alpha_2 = \pi^2/6$ according to (5).

4.2. Stress tensor of loaded plane with a linear row of circular holes

An infinite plane contains an infinite row of circular holes with radius R. The distance between the neighboring holes is l. The internal surfaces of all holes in the row are free of loading. The plane is loaded with a uniaxial stress p which makes angle ω with the Ox axis.

In order to obtain the stress tensor in the strained plane containing a linear row of holes, let us carry out the calculations according to the algorithm proposed in Section 3:

1. The complex potentials $\varphi_\infty(z)$ and $\psi_\infty(z)$ for the isolated hole in the material subjected to external loading with the constant stress p along the line making an angle ω with the Ox axis of the Cartesian coordinate system are given by [9]

$$\begin{aligned} \varphi_\infty(z) &= \frac{pz}{4}\left(1 + \frac{R^2}{z^2}2e^{2iw}\right), \\ \psi_\infty(z) &= -\frac{pz}{2}\left(e^{-2iw} + \frac{R^2}{z} - \frac{R^4}{z^4}e^{2iw}\right). \end{aligned} \tag{25}$$

The complex potentials $\varphi_0(z)$ and $\psi_0(z)$ describing the strain of the plane without holes are equal to

$$\varphi_0(z) = \frac{pz}{4}, \quad \psi_0(z) = -\frac{pz}{2}\mathrm{e}^{-2\mathrm{i}w}. \tag{26}$$

2. The zeroth order terms $K_0(z)$ and $L_0(z)$ of expressions (8) and (9) can be written as follows:

$$K_0(z) = \frac{pR^2}{2z}\mathrm{e}^{2\mathrm{i}w}, \quad L_0(z) = -\frac{pR^2}{2z}\left(1 + \left(1 - \frac{R^2}{z^2}\right)\mathrm{e}^{2\mathrm{i}w}\right). \tag{27}$$

3. The first order terms of expansions (8) and (9) can be written as

$$K_2(z) = -\frac{\alpha_2 pR^4}{z}(1 + 2\mathrm{e}^{-2\mathrm{i}w}),$$
$$L_2(z) = \frac{\alpha_2 pR^4}{z}\left(1 + 3\mathrm{e}^{-2\mathrm{i}w} + \mathrm{e}^{2\mathrm{i}w} - (1 + 2\mathrm{e}^{-2\mathrm{i}w})\frac{R^2}{z^2}\right). \tag{28}$$

4. Approximate expressions for the complex potentials $\varphi(z)$ and $\psi(z)$ are as follows:

$$\varphi(z) \approx \varphi^{(0)}(z) + \varphi^{(1)}(z)\left(\frac{R}{l}\right)^2, \quad \psi(z) \approx \psi^{(0)}(z) + \psi^{(1)}(z)\left(\frac{R}{l}\right)^2, \tag{29}$$

where

$$\varphi^{(0)}(z) = \frac{pz}{4}\left(1 + 2\mathrm{e}^{2\mathrm{i}w}\frac{R^2}{z^2}\right),$$
$$\psi^{(0)}(z) = -\frac{pz}{2}\left(\mathrm{e}^{-2\mathrm{i}w} + \frac{R^2}{z} - \frac{R^4}{z^4}\mathrm{e}^{2\mathrm{i}w}\right),$$
$$\varphi^{(1)}(z) = -\alpha_2 pz\left(\mathrm{e}^{2\mathrm{i}w} + \frac{R^2}{z^2}(1 + 2\mathrm{e}^{-2\mathrm{i}w})\right), \tag{30}$$
$$\psi^{(1)}(z) = \alpha_2 pz\left[1 + 2\mathrm{e}^{2\mathrm{i}w} + (\mathrm{e}^{-2\mathrm{i}w} + \mathrm{e}^{2\mathrm{i}w})\frac{R}{z^2} - (1 + 2\mathrm{e}^{-2\mathrm{i}w})\frac{R^4}{z^4}\right].$$

5. The approximate expression for the stress tensor $\sigma_{ij}(r,\theta), (i,j = r,\theta)$ is given by

$$\sigma_{ij}(r,\theta) \approx \sigma_{ij}^{(0)}(r,\theta) + \sigma_{ij}^{(1)}(r,\theta)\left(\frac{R}{l}\right)^2, \tag{31}$$

where the zeroth order approximation

$$\sigma_{rr}^{(0)}(r,\theta) = \frac{p}{2}\left(1 - \frac{R^2}{r^2} + \left(1 - \frac{4R^2}{r^2} + \frac{3R^4}{r^4}\right)\cos 2(\omega - \theta)\right),$$
$$\sigma_{\theta\theta}^{(0)}(r,\theta) = \frac{p}{2}\left(1 + \frac{R^2}{r^2} - \left(1 + \frac{3R^4}{r^4}\right)\cos 2(\omega - \theta)\right),$$
$$\sigma_{r\theta}^{(0)}(r,\theta) = \frac{p}{2}\left(1 + \frac{2R^2}{r^2} - \frac{3R^4}{r^4}\right)\sin 2(\omega - \theta), \tag{32}$$

coincides with that obtained by [9,13] for an isolated circular hole in the strained plane. The first order term in expansion (31) is given by

$$\sigma_{rr}^{(1)}(r,\theta) = -\alpha_2 p\left(2\cos 2\omega\left(1 - \frac{R^2}{r^2}\right) + \left(1 - \frac{4R^2}{r^2} + \frac{3R^4}{r^4}\right)(2\cos 2(\omega + \theta) + \cos 2\theta)\right),$$
$$\sigma_{\theta\theta}^{(1)}(r,\theta) = \alpha_2 p\left(-2\cos 2\omega\left(1 + \frac{R^2}{r^2}\right) + \left(1 + \frac{3R^4}{r^4}\right)(2\cos 2(\omega + \theta) + \cos 2\theta)\right),$$
$$\sigma_{r\theta}^{(1)}(r,\theta) = \alpha_2 p\left(1 + \frac{2R^2}{r^2} - \frac{3R^4}{r^4}\right) \times (2\sin 2(\omega + \theta) + \sin 2\theta). \tag{33}$$

5. Applications

The approximate expression of the actual stress tensor is written down as

$$\sigma_{ij}(r,\theta) \approx \sigma_{ij}^{(0)}(r,\theta) + \sigma_{ij}^{(1)}(r,\theta)\left(\frac{R}{l}\right)^2, \tag{34}$$

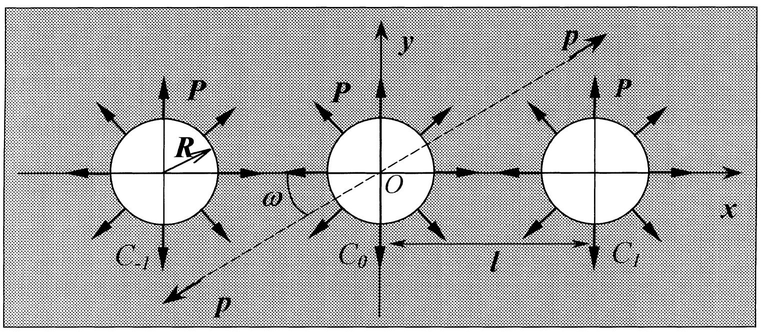

Fig. 2. Uniaxial loading of the plain with a constant stress p constituting angle ω with Ox axis of the Cartesian coordinate system.

where the factors $\sigma_{ij}^{(k)}(r,\theta)$ are evaluated for particular internal and external loadings.

The actual stress tensor can be obtained through the superposition of the external and internal loadings acting on the material. Let us apply the results obtained in the previous section for construction of stress tensors for particular stress concentrator configurations and external loadings.

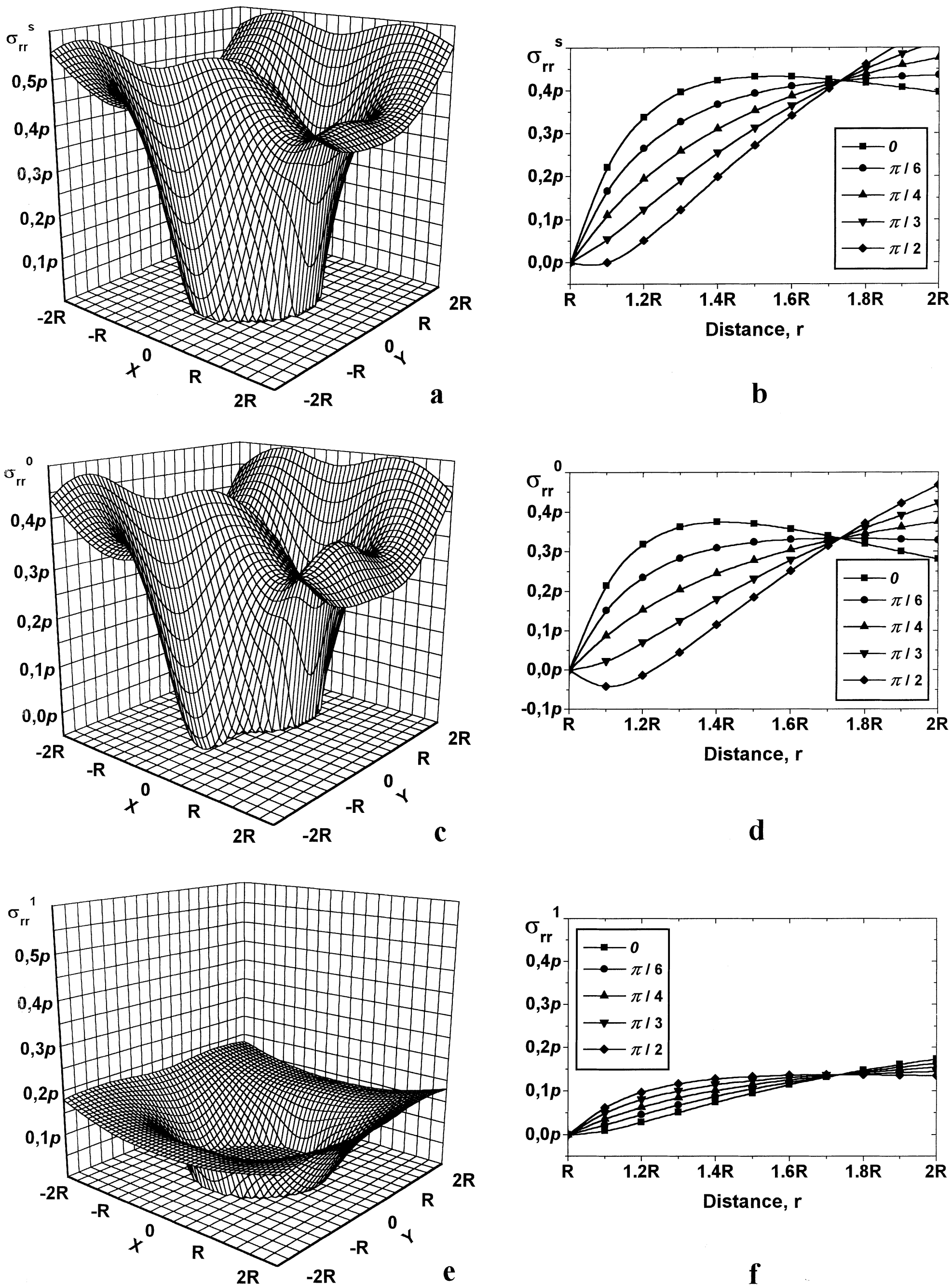

Fig. 3. Dependences of $\sigma_{rr}^{(s)}(x,y)$ (a), $\sigma_{rr}^{(0)}(x,y)$ (c) and $\sigma_{rr}^{(1)}(x,y)(R/l)^2$ (e) and respective cross-sections (b), (d), (f) for different meanings of angle θ (shown in the figure) in the case of uniaxial loading of material with a row of circular voids.

5.1. Uniaxial loading of the material with a row of circular stress concentrators

The zeroth order terms of expansion (34) in the case of uniaxial tension (see Fig. 2) of a material with a row of stress concentrators according to Eqs. (32) and (23) are given by

$$\sigma_{rr}^{(0)}(r,\theta) = \frac{p}{2}\left(1 - \left(1 + \frac{2P}{p}\right)\frac{R^2}{r^2} + \left(1 - \frac{4R^2}{r^2} + \frac{3R^4}{r^4}\right)\cos 2(\omega - \theta)\right),$$

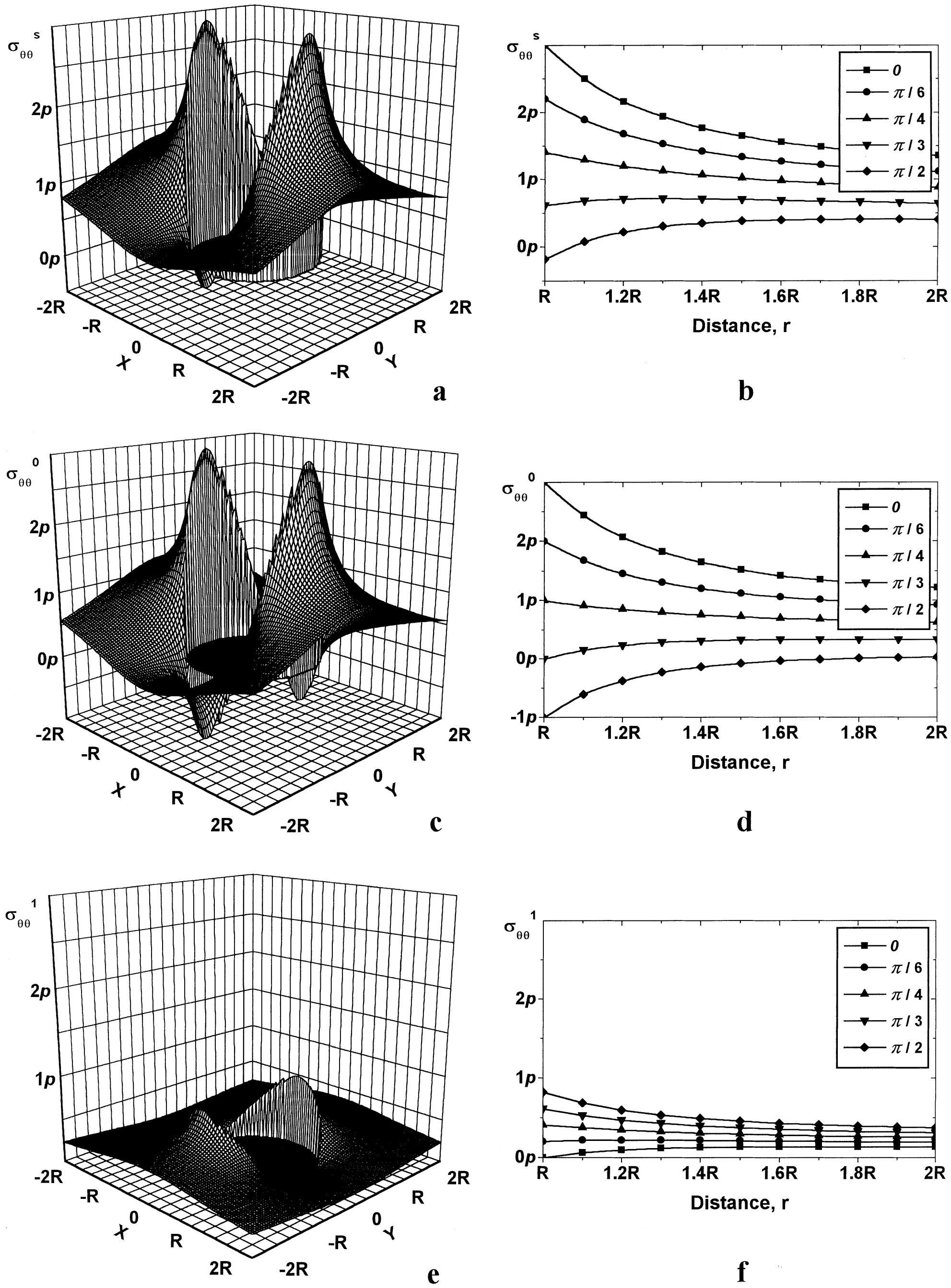

Fig. 4. Dependences of $\sigma_{\theta\theta}^{(s)}(x,y)$ (a), $\sigma_{\theta\theta}^{(0)}(x,y)$ (c) and $\sigma_{\theta\theta}^{(1)}(x,y)(R/l)^2$ (e) and respective cross-sections (b), (d), (f) for different meanings of angle θ (shown in the figure) in the case of uniaxial loading of material with a row of circular voids.

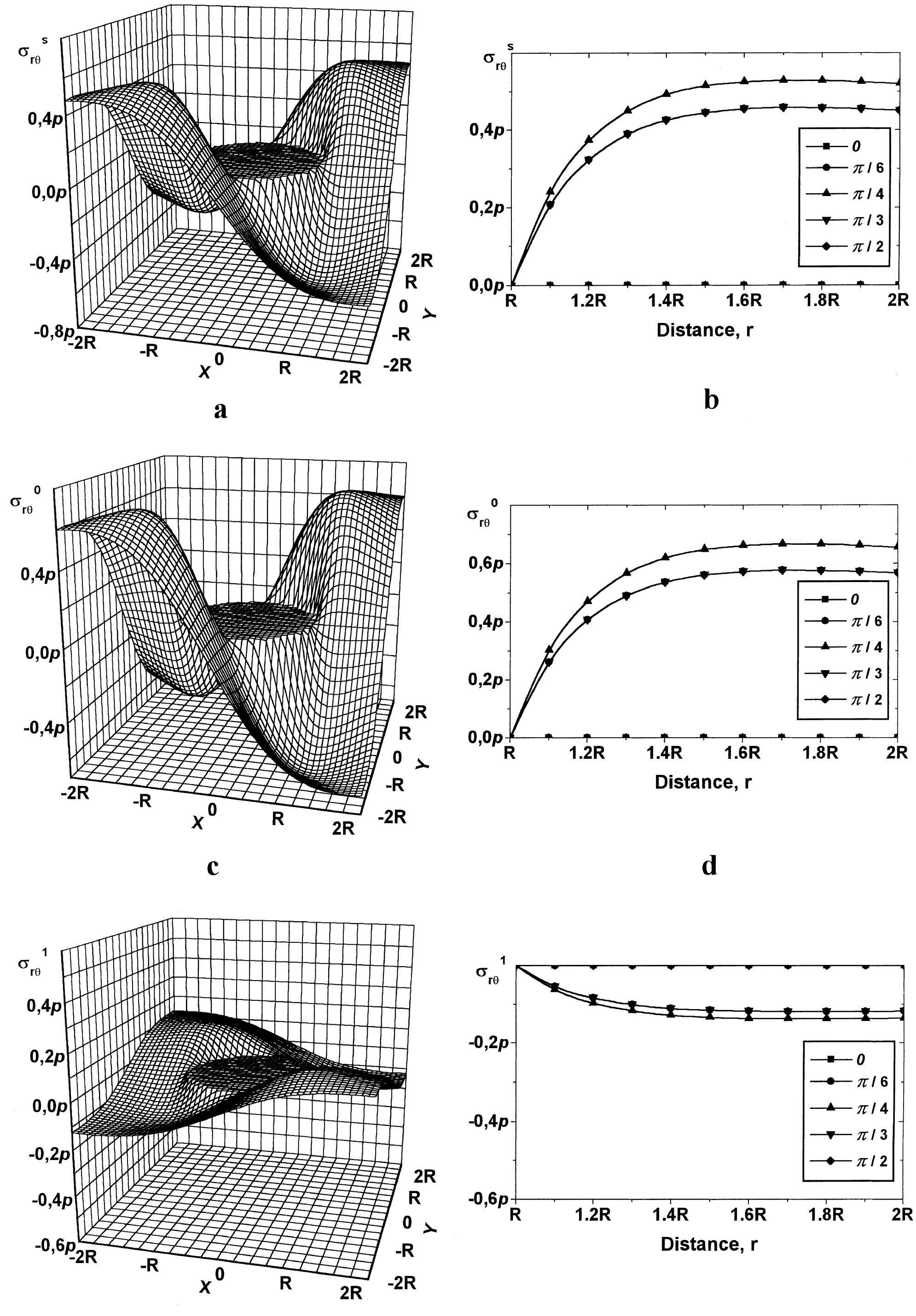

Fig. 5. Dependences of $\sigma_{r\theta}^{(s)}(x,y)$ (a), $\sigma_{r\theta}^{(0)}(x,y)$ (c) and $\sigma_{r\theta}^{(1)}(x,y)(R/l)^2$ (e) and respective cross-sections (b), (d), (f) for different meanings of angle θ (shown in the figure) in the case of uniaxial loading of material with a row of circular voids.

$$\sigma_{\theta\theta}^{(0)}(r,\theta) = \frac{p}{2}\left(1 + \left(1 + \frac{2P}{p}\right)\frac{R^2}{r^2} - \left(1 + \frac{3R^4}{r^4}\right)\cos 2(\omega - \theta)\right), \quad (35)$$
$$\sigma_{r\theta}^{(0)}(r,\theta) = \frac{p}{2}\left(1 + \frac{2R^2}{r^2} - \frac{3R^4}{r^4}\right)\sin 2(\omega - \theta),$$

and the first order terms $\sigma_{ij}^{(1)}(r,\theta)$ according to Eqs. (33) and (24) are equal to

$$\sigma_{rr}^{(1)}(r,\theta) = -\alpha_2 p\left(2\cos 2\omega\left(1 - \frac{R^2}{r^2}\right) + \left(1 - \frac{4R^2}{r^2} + \frac{3R^4}{r^4}\right) \times \left(2\cos 2(\omega+\theta) + \left(1 + \frac{2P}{p}\right)\cos 2\theta\right)\right),$$
$$\sigma_{\theta\theta}^{(1)}(r,\theta) = \alpha_2 p\left(-2\cos 2\omega\left(1 + \frac{R^2}{r^2}\right) + \left(1 + \frac{3R^4}{r^4}\right) \times \left(2\cos 2(\omega+\theta) + \left(1 + \frac{2P}{p}\right)\cos 2\theta\right)\right),$$
$$\sigma_{r\theta}^{(1)}(r,\theta) = \alpha_2 p\left(1 + \frac{2R^2}{r^2} - \frac{3R^4}{r^4}\right) \times \left(2\sin 2(\omega+\theta) + \left(1 + \frac{2P}{p}\right)\sin 2\theta\right). \quad (36)$$

Dependences of $\sigma_{ij}^{(0)}(r,\theta)$, $\sigma_{ij}^{(1)}(r,\theta)(R/l)^2$ and $\sigma_{ij}^{(S)}(r,\theta) = \sigma_{ij}^{(0)}(r,\theta) + \sigma_{ij}^{(1)}(r,\theta)(R/l)^2$ for the case of uniaxial loading of a material with a linear row of circular voids ($P = 0$) are shown in Fig. 3 (for $\sigma_{rr}(r,\theta)$), and Fig. 4 (for $\sigma_{\theta\theta}(r,\theta)$) and Fig. 5 (for $\sigma_{r\theta}(r,\theta)$). The distance l between centers of the voids is equal to $4R$. The material is loaded with constant stress p along the line perpendicular to the line of centers of voids in the row ($\omega = \pi/2$).

5.2. Uniform loading of the material with a row of circular stress concentrators

The zeroth order terms of expansions (34) in the case of uniform loading (see Fig. 6) are obtained by means of (32) and (23) in the following form:

$$\sigma_{rr}^{(0)}(r,\theta) = p\left(1 - \left(1 + \frac{P}{p}\right)\frac{R^2}{r^2}\right),$$
$$\sigma_{\theta\theta}^{(0)}(r,\theta) = p\left(1 + \left(1 + \frac{P}{p}\right)\frac{R^2}{r^2}\right), \quad (37)$$
$$\sigma_{r\theta}^{(0)}(r,\theta) = 0,$$

whereas the first order terms $\sigma_{ij}^{(1)}(r,\theta)$ according to (33) and (24) are given by

$$\sigma_{rr}^{(1)}(r,\theta) = -2\alpha_2\left(1 - \frac{4R^2}{r^2} + \frac{3R^4}{r^4}\right)(p+P)\cos 2\theta,$$
$$\sigma_{\theta\theta}^{(1)}(r,\theta) = 2\alpha_2\left(1 + \frac{3R^4}{r^4}\right)(p+P)\cos 2\theta, \quad (38)$$
$$\sigma_{r\theta}^{(1)}(r,\theta) = 2\alpha_2\left(1 + \frac{2R^2}{r^2} - \frac{3R^4}{r^4}\right)(p+P)\sin 2\theta.$$

Dependences of $\sigma_{ij}^{(0)}(r,\theta)$, $\sigma_{ij}^{(1)}(r,\theta)(R/l)^2$ and $\sigma_{ij}^{(S)}(r,\theta) = \sigma_{ij}^{(0)}(r,\theta) + \sigma_{ij}^{(1)}(r,\theta)(R/l)^2$ for the case of uniform loading of a material with linear row of circular voids ($P = 0$) are shown in Fig. 7 (for $\sigma_{rr}(r,\theta)$), Fig. 8 (for $\sigma_{\theta\theta}(r,\theta)$) and Fig. 9 (for $\sigma_{r\theta}(r,\theta)$). The distance l between centers of the voids is equal to $4R$.

5.3. Simple shear of the material with a row of circular stress concentrators

The zeroth order terms of expansions (34) in the case of simple shear of the material with row of circular stress concentrators (see Fig. 10) are obtained in the form

$$\sigma_{rr}^{(0)}(r,\theta) = p\left(1 - \frac{4R^2}{r^2} + \frac{3R^4}{r^4}\right)\cos 2(\omega - \theta) - \frac{PR^2}{r^2},$$
$$\sigma_{\theta\theta}^{(0)}(r,\theta) = p\left(1 + \frac{3R^4}{r^4}\right)\cos 2(\omega - \theta) + \frac{PR^2}{r^2}, \quad (39)$$
$$\sigma_{r\theta}^{(0)}(r,\theta) = p\left(1 + \frac{2R^2}{r^2} - \frac{3R^4}{r^4}\right)\sin 2(\omega - \theta),$$

and the first order terms $\sigma_{ij}^{(1)}(r,\theta)$ are equal to

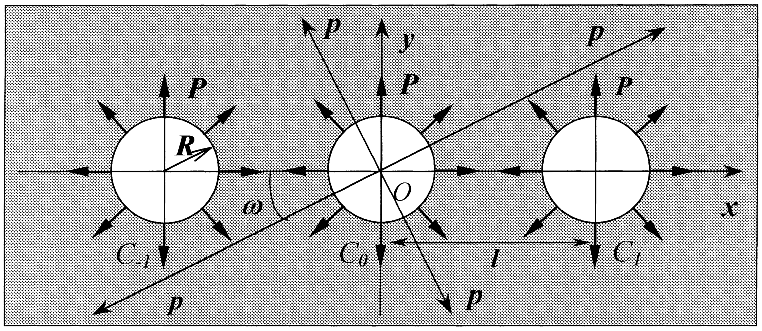

Fig. 6. Uniform loading of the plain with a row of circular stress concentrators.

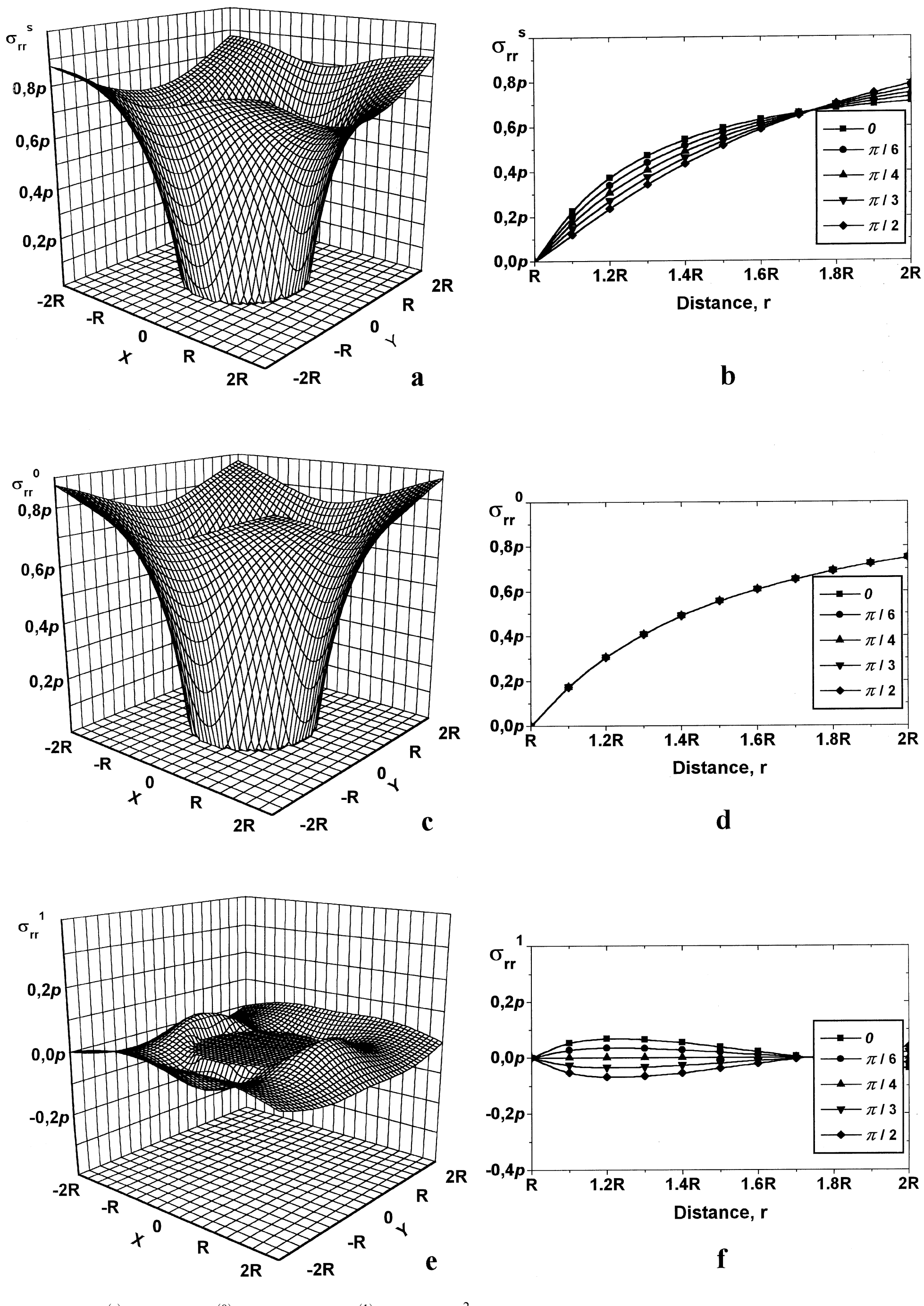

Fig. 7. Dependences of $\sigma_{rr}^{(s)}(x,y)$ (a), $\sigma_{rr}^{(0)}(x,y)$ (c) and $\sigma_{rr}^{(1)}(x,y)(R/l)^2$ (e) and respective cross-sections (b), (d), (f) for different meanings of angle θ (shown in the figure) in the case of uniform loading of material with a row of circular voids.

$$\sigma_{rr}^{(1)}(r,\theta) = -4\alpha_2 p\Bigg(\cos 2\omega\left(1-\frac{R^2}{r^2}\right)+\left(1-\frac{4R^2}{r^2}+\frac{3R^4}{r^4}\right) \times\left(\cos 2(\omega.+\theta)+\left(1+\frac{P}{2p}\right)\cos 2\theta\right)\Bigg),$$

$$\sigma_{\theta\theta}^{(1)}(r,\theta) = 4\alpha_2 p\Bigg(-\cos 2\omega\left(1+\frac{R^2}{r^2}\right)+\left(1+\frac{3R^4}{r^4}\right) \times\left(\cos 2(\omega.+\theta)+\left(1+\frac{P}{2p}\right)\cos 2\theta\right)\Bigg),$$

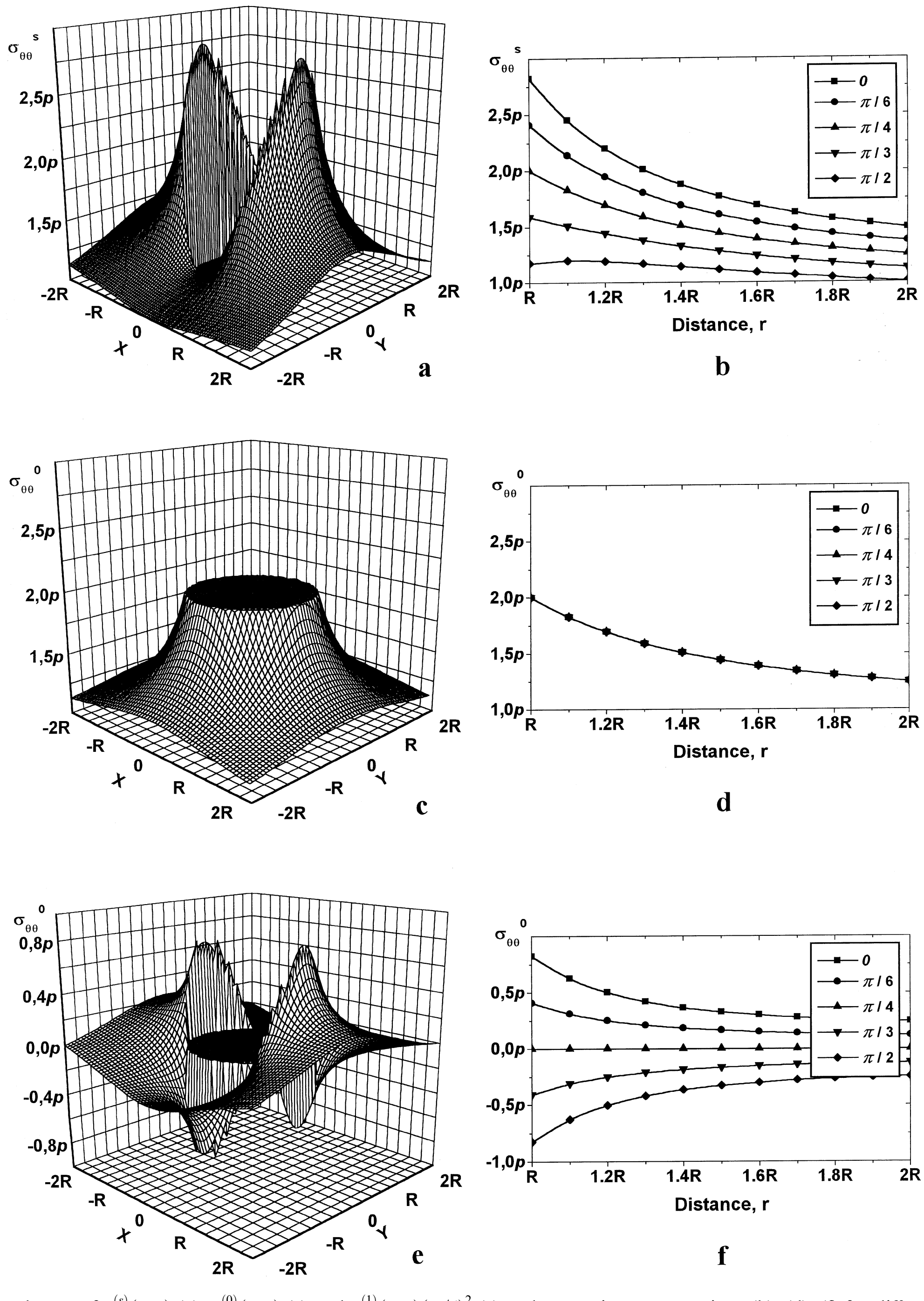

Fig. 8. Dependences of $\sigma_{\theta\theta}^{(s)}(x,y)$ (a), $\sigma_{\theta\theta}^{(0)}(x,y)$ (c) and $\sigma_{\theta\theta}^{(1)}(x,y)(R/l)^2$ (e) and respective cross-sections (b), (d), (f) for different meanings of angle θ (shown in the figure) in the case of uniaxial loading of material with a row of circular voids.

$$\sigma_{r\theta}^{(1)}(r,\theta) = 4\alpha_2 p + \left(1 + \frac{2R^2}{r^2} - \frac{3R^4}{r^4}\right) \times \left(\sin 2(\omega+\theta) + \left(1 + \frac{P}{2p}\right)\sin 2\theta\right). \quad (40)$$

Dependences of $\sigma_{ij}^{(0)}(r,\theta)$, $\sigma_{ij}^{(1)}(r,\theta)(R/l)^2$ and $\sigma_{ij}^{(S)}(r,\theta) = \sigma_{ij}^{(0)}(r,\theta) + \sigma_{ij}^{(1)}(r,\theta)(R/l)^2$ for the case of simple shear of a material with linear row of circular voids ($P = 0$) are shown in Fig. 11 (for $\sigma_{rr}(r,\theta)$),

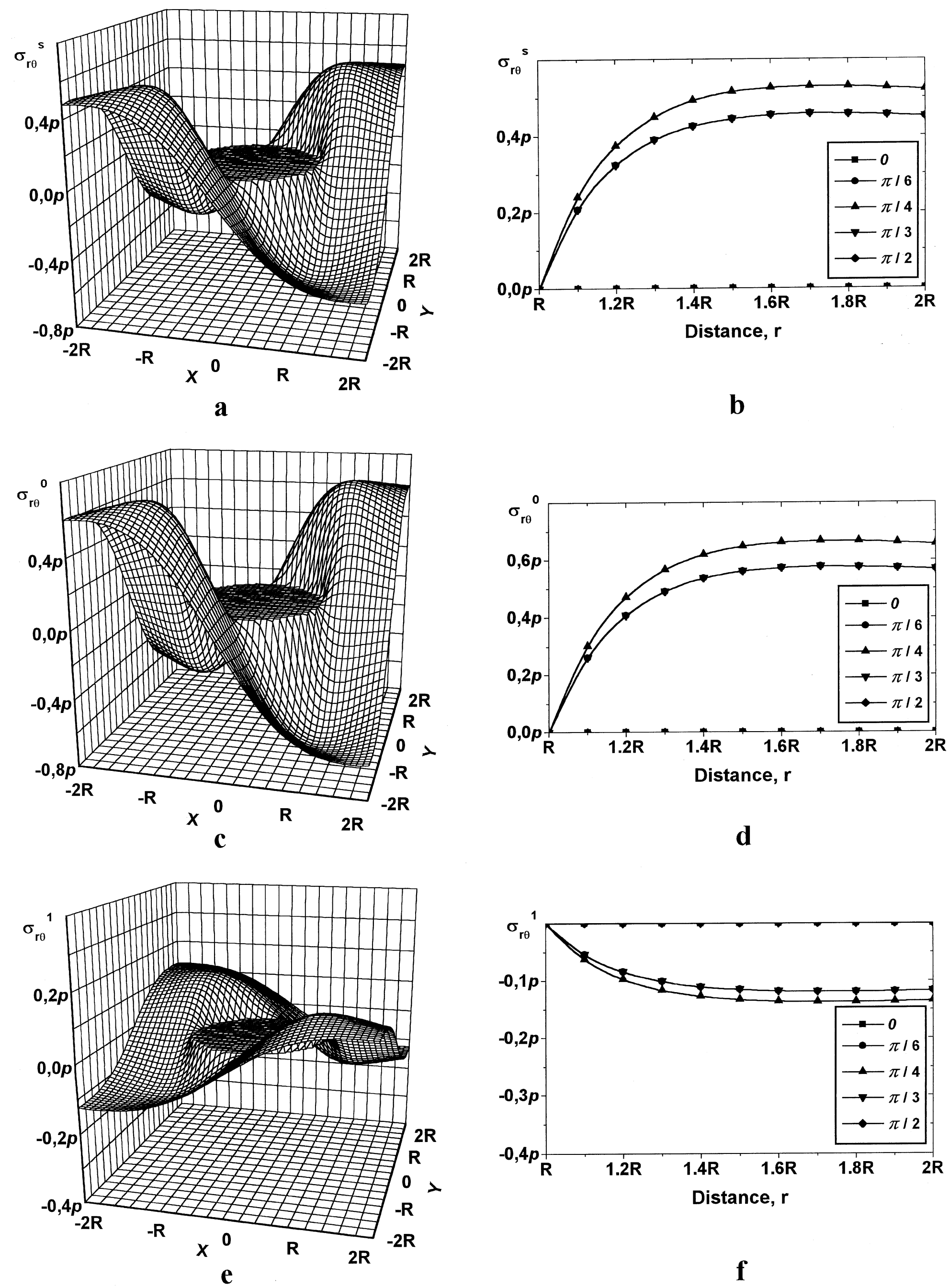

Fig. 9. Dependences of $\sigma_{r\theta}^{(s)}(x,y)$ (a), $\sigma_{r\theta}^{(0)}(x,y)$ (c) and $\sigma_{r\theta}^{(1)}(x,y)(R/l)^2$ (e) and respective cross-sections (b), (d), (f) for different meanings of angle θ (shown in the figure) in the case of uniaxial loading of material with a row of circular voids.

Fig. 12 (for $\sigma_{\theta\theta}(r,\theta)$) and Fig. 13 (for $\sigma_{r\theta}(r,\theta)$). The distance l between centers of voids in the row is equal to $4R$. The simple shear is applied along the line of centers of voids in the row ($\omega = \pi/4$, see Fig. 10).

6. Effect of a row of stress concentrators on the local stress field redistribution of a loaded material

The influence of a row of stress concentrators on the stress tensor in a material subjected to external loading

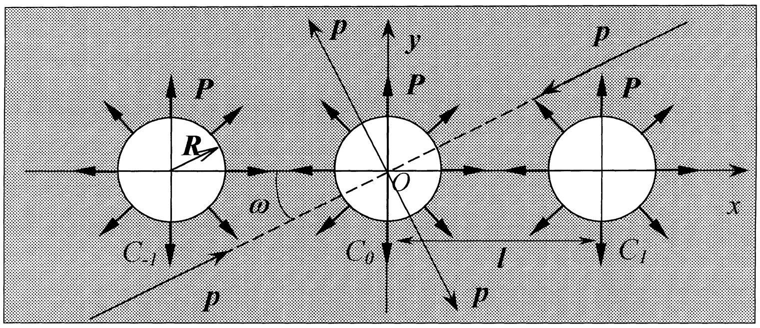

Fig. 10. Simple shear of the plain with a row of circular stress concentrators.

is investigated for the case of the material with a linear row of circular voids.

6.1. Uniaxial loading of a material with a row of circular voids

The particular case of uniaxial loading of the material with constant stress p along the line perpendicular to the line of centers of voids in the row ($\omega = \pi/2$) is interesting for practical application, namely investigation of high temperature radiation embrittlement.

Dependences of the zero order term $\sigma_{ij}^{(0)}(r,\theta)$, the first order term $\sigma_{ij}^{(1)}(r,\theta)(R/l)^2$ and their sum $\sigma_{ij}^{(S)}(r,\theta)$ for the case of uniaxial loading of a material with a row of circular voids are shown in Fig. 3 (for $\sigma_{rr}(r,\theta)$), Fig. 4 (for $\sigma_{\theta\theta}(r,\theta)$) and Fig. 5 (for $\sigma_{r\theta}(r,\theta)$). The distance l between centers of the voids is equal to $4R$.

The contribution of the row to $\sigma_{rr}^{(S)}(r,\theta)$ is of the order of the applied external stress provided $l \sim R$ (see Fig. 3). For small distances $R < r < 1.75R$ from the void center the contribution increases with increase of θ from 0 to $\pi/2$. For distances $1.75R < r < 2R$ the reverse dependence occurs. The presence of neighbor stress concentrators does not affect the general tendencies of $\sigma_{rr}(r,\theta)$, except for a small region near the void surface, where the small compressive stress reduces to a near zero value.

The maximum tensile stress $\sigma_{\theta\theta}^{(S)}(r,\theta)|_{\max} = \sigma_{\theta\theta}^{(0)}(r,\theta)|_{\max} = 3p$ is independent of the presence/absence of the row of voids (see Fig. 4) and occurs on the void surface at $\theta = 0$.

The contribution of the row increases as the angle θ increases from 0 to $\pi/2$. The near surface region with compressive stress reduces significantly due to the presence of the row (see Fig. 4).

The tangential stress $\sigma_{r\theta}^{(S)}(r,\theta)$, either reduces (in comparison with the isolated void) or becomes invariant (for $\theta = \pi/4$, where $\sigma_{r\theta}^{(S)}(r,\theta) = \sigma_{r\theta}^{(0)}(r,\theta) = 0$) due to the presence of the row of voids (see Fig. 5).

6.2. Uniform loading of a material with a row of circular voids

In the case of uniform loading, the presence of the row of voids results in anisotropy of the components σ_{rr} and $\sigma_{\theta\theta}$ of the stress tensor. These components are independent of the angle θ in the zeroth order approximation. However, taking into consideration the first order terms leads to the appearance of angular dependence of the components σ_{rr} and $\sigma_{\theta\theta}$ of the stress tensor. The contribution of the row to the $\sigma_{rr}^{(S)}(r,\theta)$ is absent for $\theta = \pi/4$, negative for $\pi/4 < \theta < \pi/2$ and positive for $0 < \theta < \pi/4$ provided the distance from the void center falls into the region $R < r < 1.75R$. Otherwise, the angular dependence of the contribution changes its sign (see Fig. 7). A similar general angular dependence of the contribution of the row to $\sigma_{\theta\theta}^{(S)}(r,\theta)$ occurs, but its sign is invariant (see Fig. 8). The maximum tensile stress $\sigma_{\theta\theta}^{(S)}(r,\theta)$ increases from $2p$ (for the isolated void) up to $2(1 + 2\pi^2/3)p$ (for the case $l \to 2R$) due to the presence of the row of voids.

The tangential stress $\sigma_{r\theta}^{(S)}(r,\theta)$ either reduces (in comparison with the isolated void) or becomes invariant (for $\theta = 0, \pi/2$) due to the presence of the row of the voids (see Fig. 9).

6.3. Simple shear of a material with a row of circular voids

The material is subjected to simple shear along the line of the centers of voids in the row.

The presence of a row of circular voids leads to a shift of both the local minimum and maximum of $\sigma_{rr}(r,\theta)$ and increases their values. For the zeroth order approximation the maximum tensile/compressive stress $\sigma_{rr}^{(0)}(r,\theta) = \pm p/3$ is achieved at $\theta = \mp\pi/4$. However due to the row of voids the local extremums of $\sigma_{rr}^{(S)}(r,\theta)$ shift counterclockwise (see Fig. 11). The shift increases with decrease of the distance l between the centers of the neighboring voids.

The same angular shift of both maximum and minimum stress of $\sigma_{\theta\theta}(r,\theta)$ on the surface of a void in the

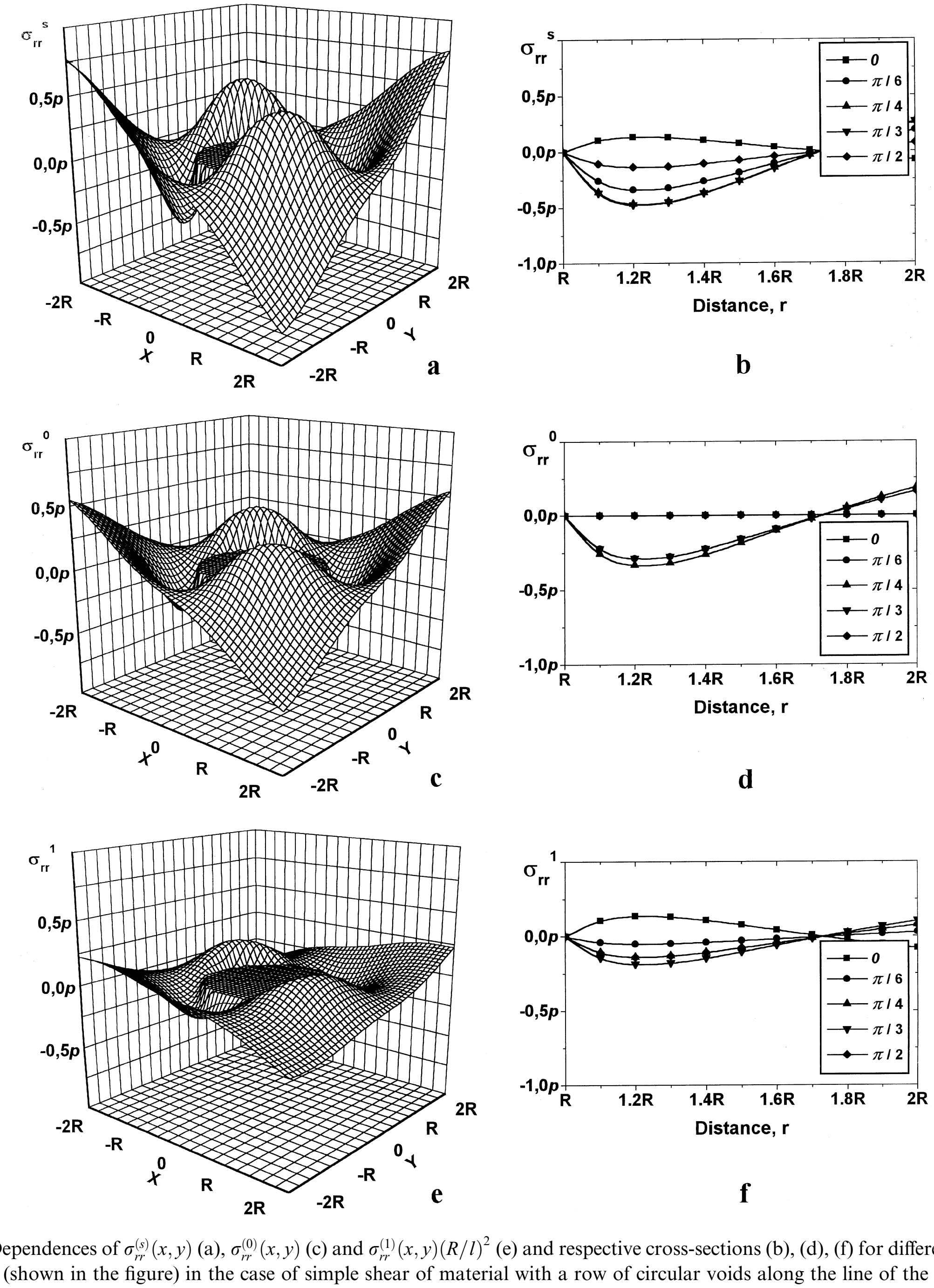

Fig. 11. Dependences of $\sigma_{rr}^{(s)}(x, y)$ (a), $\sigma_{rr}^{(0)}(x, y)$ (c) and $\sigma_{rr}^{(1)}(x, y)(R/l)^2$ (e) and respective cross-sections (b), (d), (f) for different meanings of angle θ (shown in the figure) in the case of simple shear of material with a row of circular voids along the line of the void centers.

row occurs. However, in contrast to $\sigma_{rr}(r, \theta)$ the presence of the row of voids results in reduction of actual stress $\sigma_{\theta\theta}^{(S)}(r, \theta)$ (see Fig. 12).

Due to the presence of the row of voids the absolute value of tangential stress $\sigma_{r\theta}^{(S)}(r, \theta)$ is reduced. Both maximum and minimum of $\sigma_{r\theta}^{(S)}(r, \theta)$ shift counterclockwise in comparison with those of $\sigma_{r\theta}^{(0)}(r, \theta)$ (see Fig. 13). The shift increases with decrease of the distance l between the centers of the neighbor voids.

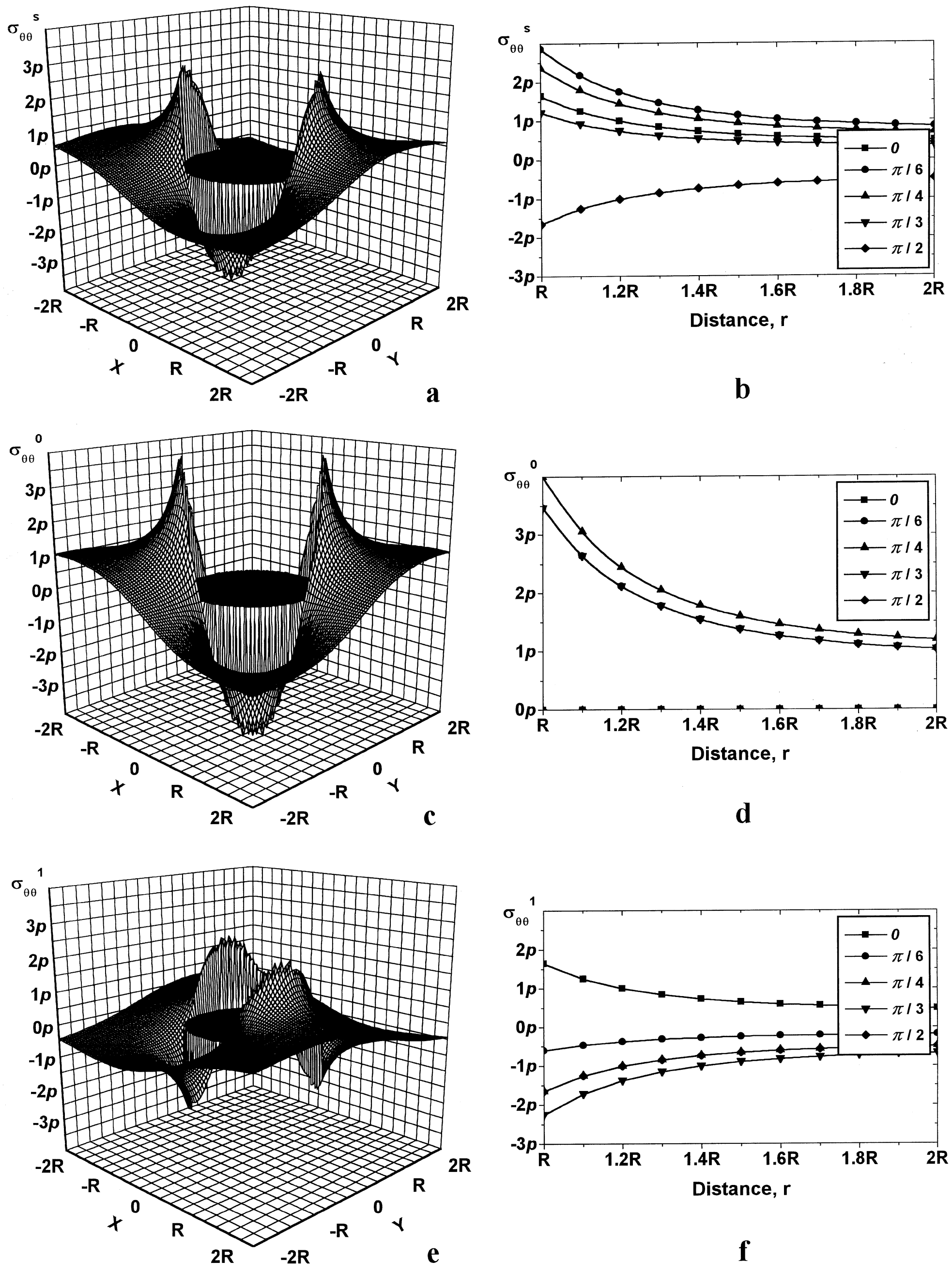

Fig. 12. Dependences of $\sigma_{\theta\theta}^{(s)}(x,y)$ (a), $\sigma_{(\theta\theta)}^{(0)}(x,y)$ (c) and $\sigma_{\theta\theta}^{(1)}(x,y)(R/l)^2$ (e) and respective cross-sections (b), (d), (f) for different meanings of angle θ (shown in the figure) in the case of simple shear of material with a row of circular voids along the line of the void centers.

7. Conclusions

The algorithm for the evaluation of the stress tensor of a loaded material with a row of stress concentrators is discussed. Stress tensors can be obtained for the rows of secondary phase precipitates, gas bubbles and voids whose shapes conformally are mapped to the unit circle by a rational function. The approach was applied to the calculation of zeroth and first order terms of the expansion of the stress tensor of a strained material with a row of

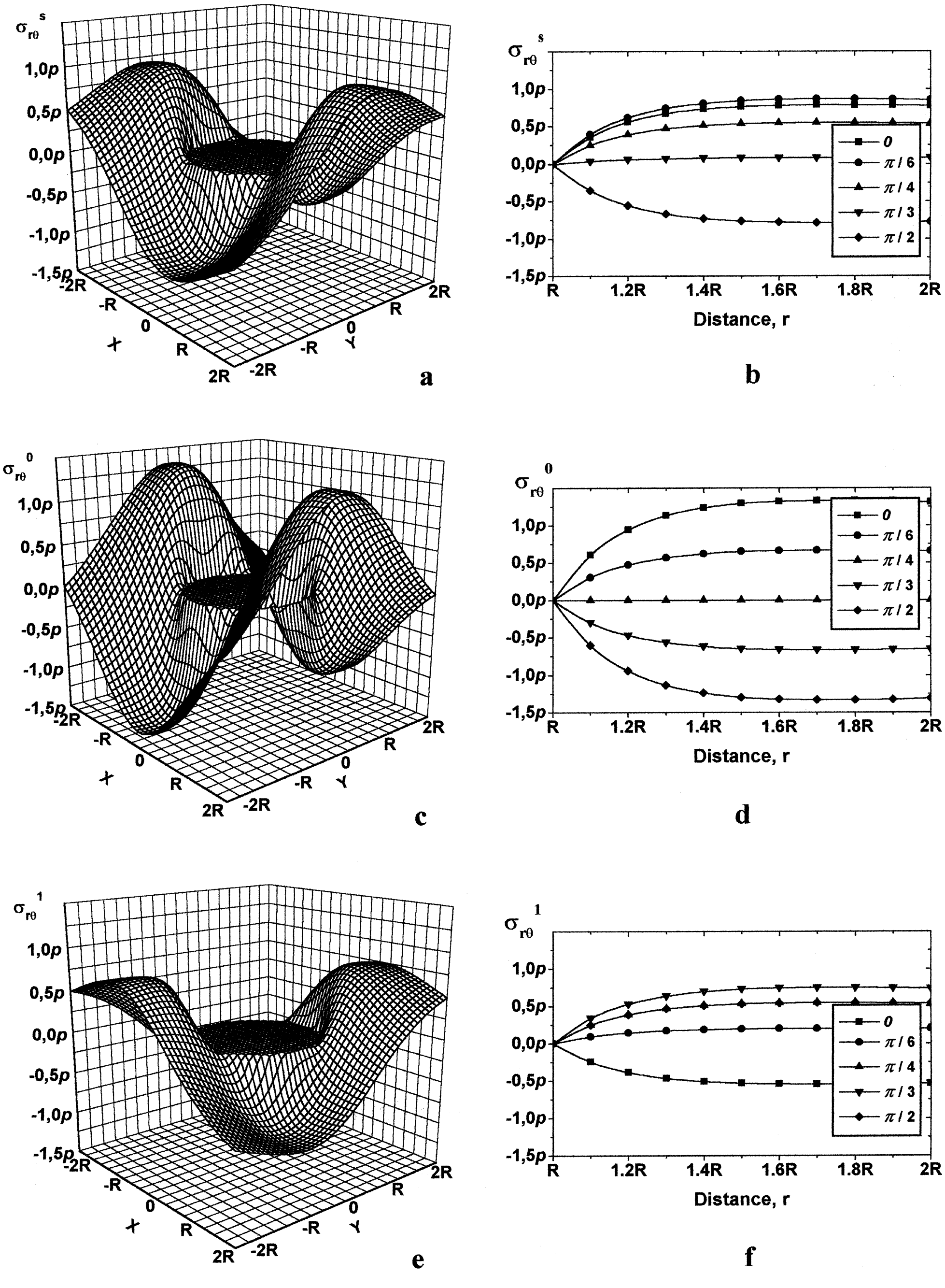

Fig. 13. Dependences of $\sigma_{r\theta}^{(s)}(x,y)$ (a), $\sigma_{r\theta}^{(0)}(x,y)$ (c) and $\sigma_{r\theta}^{(1)}(x,y)(R/l)^2$ (e) and respective cross-sections (b), (d), (f) for different meanings of angle θ (shown in the figure) in the case of simple shear of material with a row of circular voids along the line of the void centers.

circular stress concentrators over the R^2/l^2 series. Stress tensor for the following loading modes were calculated:

1. Uniaxial loading of a material with a circular row of stress concentrators.
2. Uniform loading of a material with a circular row of stress concentrators.
3. Simple shear of a material with a circular row of stress concentrators.

The results obtained and their linear combinations can be applied for evaluation of stress tensors of a number of practical applications relevant to internal and external loading within plane strain.

The effect that a row of circular voids has on the stress field redistribution was investigated. It is found that the influence is strongly dependent on the particular type of applied stress.

Obtained results will be used for evaluation of fracture toughness of a strained material with linear row of stress concentrators (in a separate paper).

References

[1] Y. Murakami, Stress Intensity Factors Handbook, vol. 1, Pergamon, Oxford, 1987, p. 241.
[2] J.C. Newman, NASA TN D 6376 (1971) 1–45.
[3] J. Tweed, D.P. Roohe, J. Eng. Sci. 11 (1973) 1185.
[4] J. Schijve, Int. J. Fracture 23 (1983) r111.
[5] O.L. Bowie, J. Math. Phys. 35 (1) (1956) 60.
[6] R.E. Voskoboinikov, J. Nucl. Mater. 270 (1999) 309.
[7] R.C. Howland, Proc. Roy. Soc. A 148 (1935) 471.
[8] G.N. Savin, DAN USSR 23 (1939) 515 (in Russian).
[9] N.I. Muskhelishvili, Some Basic Problems of the Mathematical Theory of Elasticity, P. Noordhoff, Groningen, 1953.
[10] S.G. Mikhlin, Integral equations, Gostekhizdat, 1949 (in Russian).
[11] I.I. Vorovich, A.S. Kosmodamianski, Izvestiya Academii Nauk USSR 4 (1959) 69 (in Russian).
[12] L.D. Landau, E.M. Livshits, Theory of Elasticity, Pergamon, Oxford, 1969.
[13] A.I. Lurie, Theory of Elasticity, Nauka, Moscow, 1970 (in Russian).